جامعة الحكمة إلورن نيجيريا

AL-HIKMAH UNIVERSITY, ILORIN
FACULTY OF MANAGEMENT SCIENCES
DEPARTMENT OF BANKING AND FINANCE
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LECTURE NOTE

COURSE CODE: BAN 206 **CREDIT UNIT: 3**

COURSE TITLE: MATHEMATICS FOR FINANCE II

FUNCTIONS, GRAPHS AND EQUATIONS

FUNCTIONS:

This occurs when a definite relationship exists between two variables. If x and y are two variables, then y is said to be a function of x. Moreso, if there is a relationship between x and y which ensures that the value of y can be determined for each value of x. Then, the relationship takes the form of:

$Y = 3x + 2.$

Where x ranges from -2 to +4.

Therefore, the value of y can be determined as follows:

When $x = -2$

$Y = 3(-2) + 2$

$= -6 + 2$

$\therefore y = -1$

When $x = 0$

$Y = 3(0) + 2$

$= 0 + 2$

$\therefore y = 2$

When $x = 1$

$Y = 3(1) + 2$

$= 3 + 2$

$\therefore y = 5$

When $x = 2$

$Y = 3(2) + 2$

$= 6 + 2$

$\therefore y = 8$

When $x = 3$

$Y = 3(3) + 2$

$= 9 + 2$

$\therefore y = 11$

When $x = 4$

$Y = 3(4) + 2$

$= 12 + 2$

$\therefore y = 14$

Practice Questions:

1) $Y = 2x^2 – 4x + 10$

2) $Y = 4x^2 – 12x + 16$

3) $Y = 3x^4 + 2x^2 – 10$

4) $Y = 2x^2 – 5x + 4$

5) $Y = (4 + x) (x – 1)$

6) $Y = 8x – 4 + \frac{2}{x}$

GRAPHS

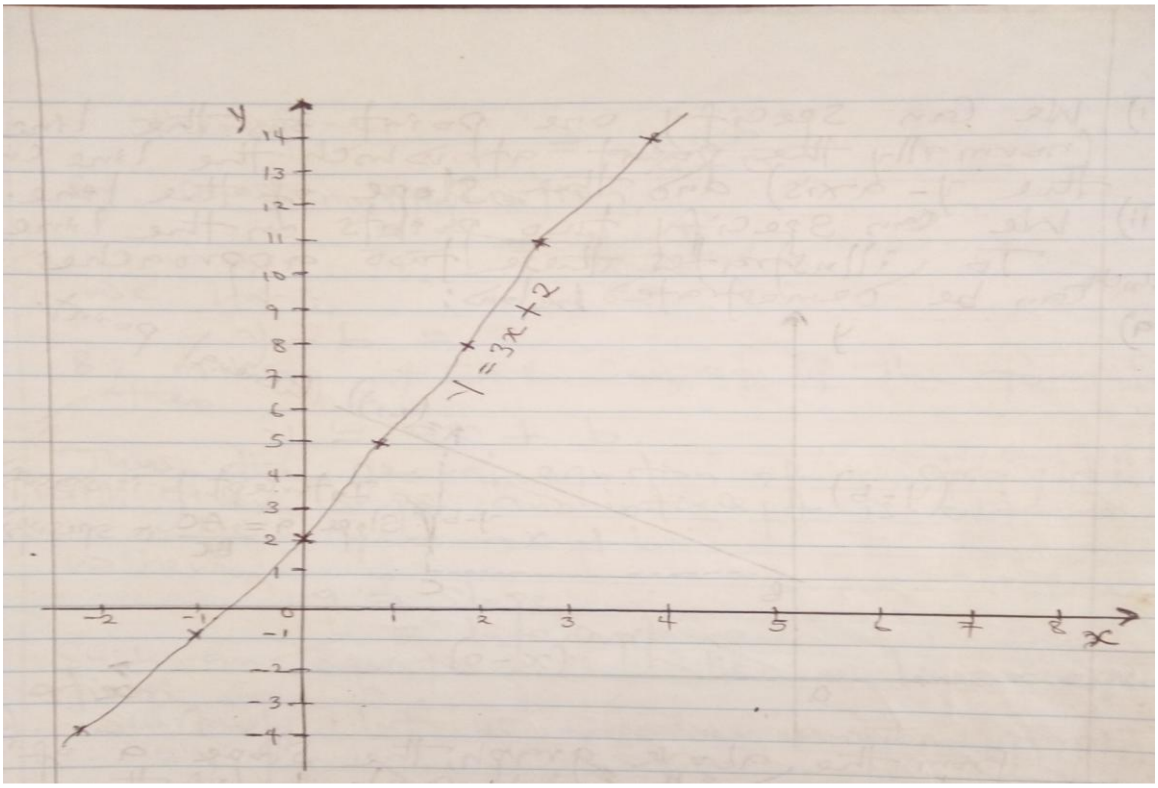
A graph is a pictorial presentation of the relationship between variables. There are many types of graphs employed in statistics depending on the nature of the data involved or the purpose for which the graph is intended. These include bar graph, pie graph, ogive, pictogram e.t.c. It is important to note that graphs provide a useful way of visualizing such relationships that exists between two varigles. For instance, in the above functions i.e y = 3x + 2, when is to be plotted in a graph, a straight line graph will be obtained.

Therefore, to plot the graph, we produce a table of values as follows:

X	-2	-1	0	1	2	3	4
y=3x+2	-4	-1	2	5	8	11	14

STRAIGHT-LINE GRAPHS

When the function y=3x + 2 was plotted in a graph, a straight-line graph was obtained. We can observe that as x increases, so also y increases.



EQUATION

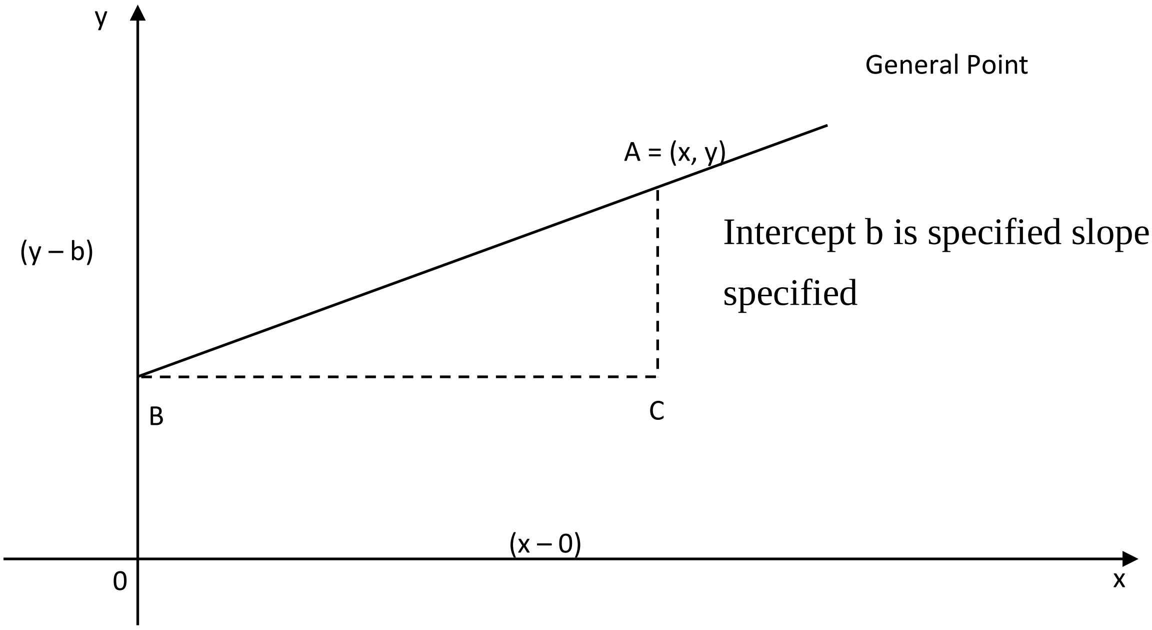
Since the graph is a straight-line, we will appreciate that, it is only necessary to plot two points through which the line can be drawn. Infact, in any equation, we can show that y = ax + b

Where a, b are constants, has a straight-line graph and conversely any straight-line graph corresponds to an equation of this form.

Moreso, any straight-line can be specified in one of two ways:

i. We can specify one point on the line (normally the point at which the line cuts the y-axis) and the slope of the line.
ii. We can specify two points on the line.

To illustrates these two approaches which can be demonstrated below:



Intercept b is specified slope a = $\frac{AC}{BC}$ is specified

From the above graph, the slope a of the line and the intercept b of the line with the y- axis are specified, where: a = $\frac{increase\ in\ y}{increase\ in\ x}$

It is important to note that measured between any two points on the line, is always the same.

And b is the value of y at which the line cuts the y-axis i.e

Point A = (x, y) and point B = (0, b). Thus,

a = $\frac{increase\ in\ y}{increase\ in\ x} = \frac{y-b}{x-0}$

therefore, a = $\frac{y-b}{x-0}$

by cross multiplication,

$\frac{a}{1} = \frac{y-b}{x-0}$

We have y – b = ax

By making y the subject of the formula, then

$y = ax + b$

Thus, the general equation of a straight-line with given slope and intercept is obtained.

i.e $y = ax + b$

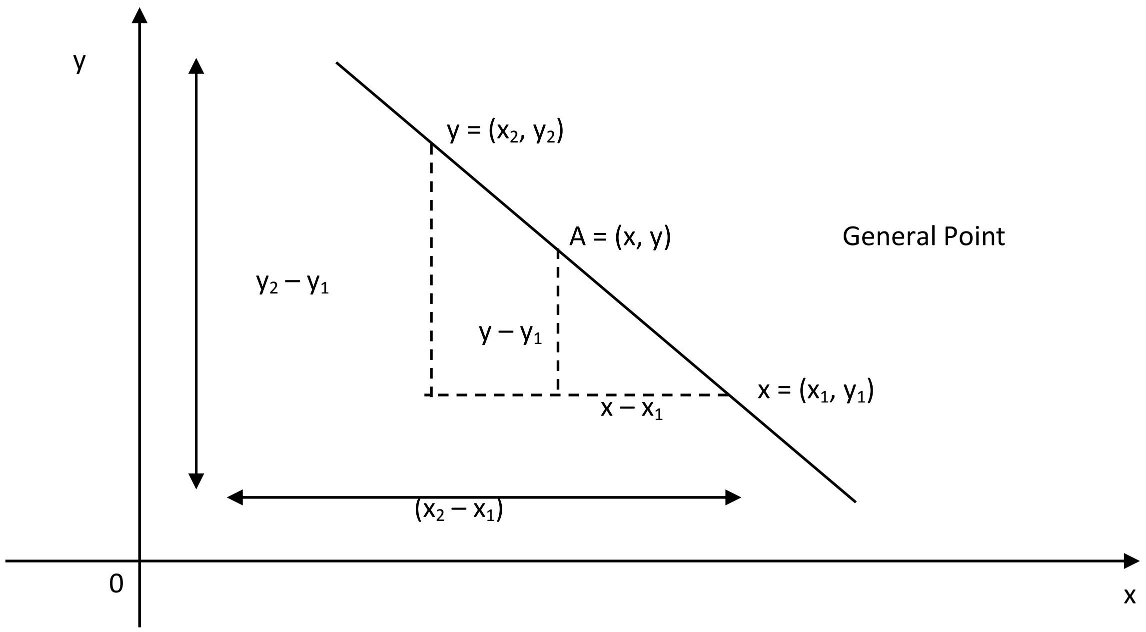
where:

a = slope

b = intercept

Such an equation is called a linear equation.

Similarly, let us consider another graph:



From the above graph, we have two given points on the line where:

$x = (x_1, y_1)$ and $y = (x_2, y_2)$

Thus, the slope measured between x and A = the slope measured between x and y. Therefore, we have:

The General equation of the straight-line through which two points (x_1, y_1) and (x_2, y_2). That is, $\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$

<u>Illustration:</u>

The amount x units of a commodity purchased daily by consumers is dependent upon the selling price (₦ per unit) and it has been observed that 20 units are demanded daily at a price of ₦300 per unit, whereas 30 units are demanded daily at a price of ₦250 per units. Assuming that the relationship between p and x is linear:

i. Find the equation for p in terms of x
ii. Find the value of x when p (price) is ₦180 per unit and draw the graph of the equation.

Solution:

i. The linear equation is given as:

$$\frac{p - p_1}{x - x_1} = \frac{p_2 - p_1}{x_2 - x_1}$$

Thus,

x_1 = 20 units, p_1 = ₦300 and,

x_2 = 30 units, p_2 = ₦250

by substitution,

$$\frac{p - 300}{x - 20} = \frac{250 - 300}{30 - 20}$$

Therefore,

$\frac{p-300}{x-20} = \frac{-50}{10}$

$= \frac{p-300}{x-20} = \frac{-5}{1}$

By cross multiplication,

p - 300 = -5(x – 20)

p – 300 = -5x + 100

by making p the subject of the formular

p = -5x + 100 + 300

= -5x + 400

∴ p = 400 - 5x

ii. The value of x when price is ₦180 per unit:

Since, p = 400 – 5x

Therefore, when p = ₦180

Thus, 180 = 400 – 5x

By collecting the like terms:

180 – 400 = -5x

-220 = -5x

By multiplying or dividing both sides by negative

220 = 5x

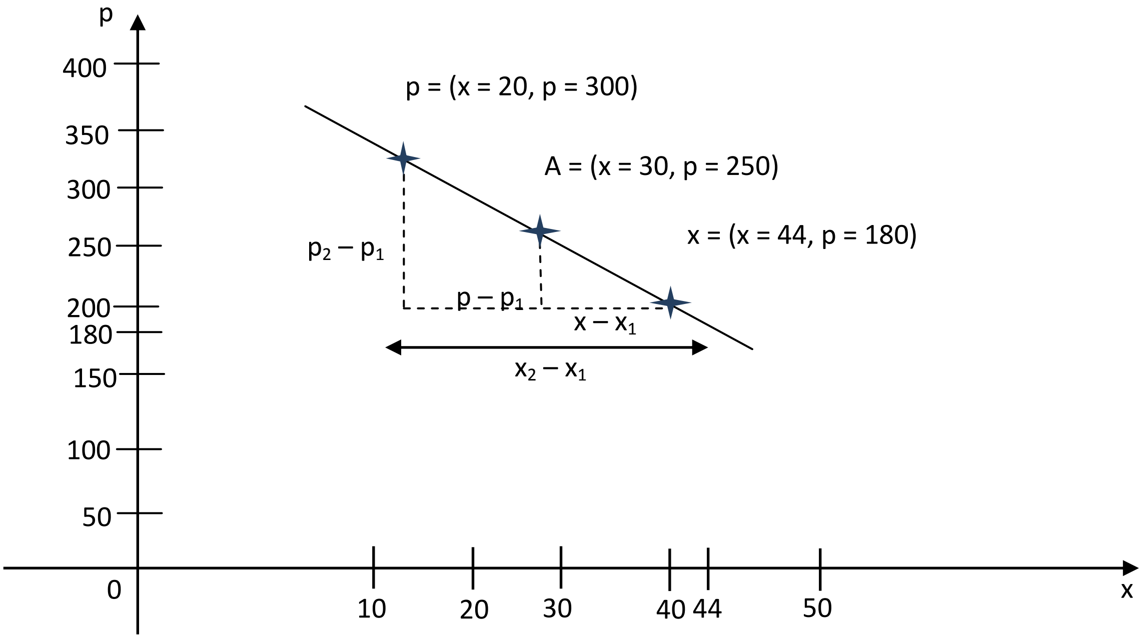
By dividing both sides by 5

$\frac{220}{5} = \frac{5x}{5}$

∴ x = 44 units

To plot the graph, we have to produce a table of values as follows:

X	20	30	44
p = 400 – 5x	300	250	180



Practice Questions:

1) The amount x unites of a commodity purchased weekly by consumers is dependent upon the selling price ₦ per unit, and it is assumed that the relationship between p and x is linear.Find the equation relating p to x for each of the following sets of observed data; and draw the graph of the equation: a)

Price per unit	150	60
x (No. of units demanded weekly)	50	80

b)

Price per unit	135	90
x (No. of units demanded weekly)	10	15

2) The amount of x units of a commodity purchased monthly by consumers is dependent upon the selling price (₦/unit), and it is assumed that the relationship between p and x is linear.

It has been observed that 500 units of commodities are demanded monthly at a price of ₦12,000 per unit, whereas 800 units are demanded monthly at a price ₦10,000 per units.

i. Find the equation for p in terms of x.
ii. Find the value of x when p (price) is ₦15,000.

SIMPLEX ALGORITHM

The simplex algorithm provides a numerical approach for handling three or more products which lends itself to computer implementation and allows any number of products to be considered. Moreso, it can be applied to both maximization and minimization problem.

MAXIMISATION PROBLEMS

Illustration:

To maximize: G = 30x + 40y profit subject to the constraints:

$3x + 6y \leq 2400$ Material 1

$5x + 8y \leq 4000$ Material 2

$3x + 3y \leq 1800$ Time (Hours)

$y \leq 300$ Demand

$x \geq 0, y \geq 0$ Additional

The algorithm is based upon the conversion of constraint inequalities to equations by introducing new variables called Slack Variables, which are traditionally denoted by S_1, S_2, e.t.c. One essential property of S_1 is that, it is non-negative, which means that S_1 is zero or positive. The other essential property is that when S_1 is added to the total materials required for x and y, this equates to the maximum available materials required.

i.e $3x + 6y + S_1 = 2400$

Slack Variables S_2, S_3 and S_4 are added for the other constraints in a similar way, since the second and third constraints are also concerned with maximum resources, and the fourth constraint deals with maximum demand.

The steps in using simplex algorithm is as follows:

Step 1:

Re-state the linear program, by converting all the constraint inequalities to equations using slack variables.

Step 2:

Write down the initial tableau and the initial feasible solution.

Step 3:

Decide which zero variable (if any) to increase, and by how much it can be increased.

The chosen variable must correspond to a negative coefficient in the f- row. One approach is to choose the variable or a variable corresponding to the numerically largest negative f-row coefficient.

For the chosen variable, each positive number in its column is divided into the corresponding quantity value in the right-hand column. The number (or a number) for which the result of the division is smallest is called the pivot. The row containing the pivot is divided by the pivot (unless the pivot is 1) to obtain the master row.

Step 4:

Use the master row to convert each number in the column for the variable chosen in step 3, other than the pivot, to zero.

Step 5:

Create a new tableau by using the master row and the new rows obtained from the row operations in step 4.

NOTE:

Steps 3, 4 and 5 are now repeated until the solution is optimum. This will be so when every coefficient in the f-row is non-negative.

Rather than choosing the variable in step 3 for which the negative f-row coefficient is numerically largest, it is often more efficient to choose the variable which actually produces the largest total increase in the value of the objective function.

Procedures in solving simplex algorithm include the following:

1. Formulation of objective function
2. Formulation of constraints
3. Standardization of the above equations
4. Using the standardised equation to form a starting tableau.
5. Recognition of the most negative value on the objective row which would be the entrying column.
6. Recognition of the least ratio and making the corresponding row as the master row.

7. The intersection of the entrying column and the master row is the pivot value (pivot element).

<u>Second Tableau:</u>

<u>Step 1:</u>

New pivot equation: This is being formed by dividing each element of the former pivot equation by the pivot value.

<u>Step 2:</u>

All other equations including the objective equation would be:

Old Equation – $\left(\begin{matrix} entrying \\ column \\ element \end{matrix}\right)$ x $\left(\begin{matrix} new\ pivot \\ equation \end{matrix}\right)$

Solution to the Illustration:

Objective Function:

To maximize: F = 30x + 40y

Subject to the constraints: $3x + 6y \leq 2{,}400$

$5x + 8y \leq 4{,}000$

$3x + 3y \leq 1{,}800$

$y \leq 300$

$x \geq 0, y \geq 0$

By re-writing the inequalities to equality equation using slack variable we therefore, have:

Objective Function:

To maximize F = 30x + 40y

Subject to the constraints: 3x + 6y = 2,400

5x + 8y = 4,000

3x + 3y = 1,800

0 + y = 300

Thus,

$x \geq 0, y \geq 0, S_1 \geq 0, S_2 \geq 0, S_3 \geq 0, S_4 \geq 0.$

(a) By solving this equation graphically:

i. 3x + 6y = 2,400

When x = 0

6y = 2,400

y = 400

similarly, when y = 0

3x = 2,400

x = 800

ii. 5x + 8y = 4,000

When x = 0

8y = 4,000

y = 500

similarly, when y = 0

5x = 4,000

x = 800

iii. 3x + 3y = 1,800

When x = 0

3y = 1,800

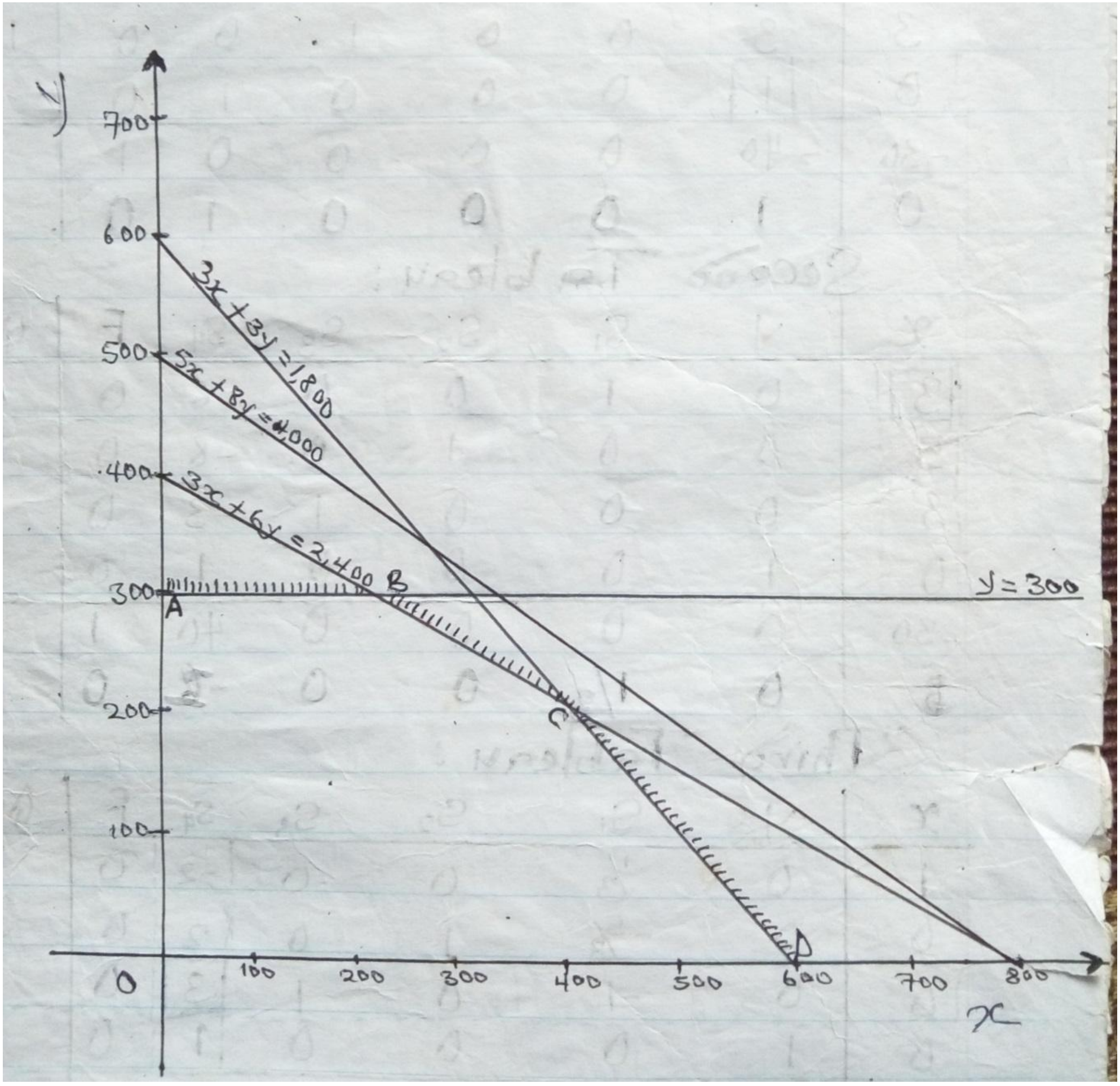
y = 600

similarly, when y = 0

3x = 1,800

x = 600

iv. y = 300, when x = 0



(b) By solving the inequalities by simplex algorithm i.e Tableau:

Objective Function:

To maximize: F = 30x + 40y => F – 30x – 40y = 0

Subject to the constraints: $3x + 6y + S_1 = 2{,}400$

$5x + 8y + S_2 = 4,000$

$3x + 3y + S_3 = 1,800$

$0 + y + S_4 = 300$

Thus,

$x \geq 0, y \geq 0, S_1 \geq 0, S_2 \geq 0, S_3 \geq 0, S_4 \geq 0.$

Tableau:

X	Y	S_1	S_2	S_3	S_4	F	Quantity	Ratio
3	6	1	0	0	0	0	2,400	400
5	8	0	1	0	0	0	4,000	500
3	3	0	0	1	0	0	1,800	600
0	1	0	0	0	1	0	300	300
-30	-40	0	0	0	0	1	0	
0	1	0	0	0	1	0	300	Master Row

Second Tableau:

X	Y	S_1	S_2	S_3	S_4	F	Quantity	Ratio
3	0	1	0	0	-6	0	600	200
5	0	0	1	0	-8	0	1600	320
3	0	0	0	1	-3	0	900	Fixed
0	1	0	0	0	1	0	300	Undefined
-30	0	0	0	0	40	1	12,000	
1	0	$\frac{1}{3}$	0	0	-2	0	200	Master Row

Third Tableau:

X	y	S_1	S_2	S_3	S_4	F	Quantity	Ratio
1	0	$\frac{1}{3}$	0	0	-2	0	200	-100
0	0	$\frac{-5}{3}$	1	0	2	0	600	300
0	0	-1	0	1	3	0	300	100
0	1	0	0	0	1	0	300	300
0	0	10	0	0	-20	1	18,000	
0	0	$\frac{-1}{3}$	0	$\frac{1}{3}$	1	0	100	Master Row

Fourth Tableau:

x	y	S_1	S_2	S_3	S_4	F	Quantity	Ratio
1	0	$\frac{-1}{3}$	0	$\frac{2}{3}$	0	0	400	-1,200
0	0	-1	1	$\frac{-2}{3}$	0	0	400	-400
0	0	$\frac{-1}{3}$	0	$\frac{1}{3}$	1	0	100	-300
0	1	$\frac{1}{3}$	0	$\frac{-1}{3}$	0	0	200	600
0	0	$\frac{10}{3}$	0	$\frac{20}{3}$	0	1	20,000	

First Tableau:

Row 1

3 – (6 x 0)

3- 0 = 3

6 – (6 x 1)

6 – 6 = 0

1 – (6 x 0)

1 – 0 = 1

0 – (6 x 0)

0 – 0 = 0

0 – (6 x 0)

0 – 0 = 0

0 – (6 x 1)

0 – 6 = -6

0 – (6 x 0)

0 – 0 = 0

2,400 – (6 x 300)

2,400 – 1800

= 600

Row 2

5 – (8 x 0)

5 – 0 = 5

8 – (8 x 1)

8 – 8 = 0

0 – (8 x 0)

0 – 0 = 0

1 – (8 x 0)

1 – 0 = 1

0 – (8 x 0)

0 – 0 = 0

0 – (8 x 1)

0 – 8 = -8

0 – (8 x 0)

0 – 0 = 0

4,000 – (8 x 300)

4,000 – 2,400

= 1,600

Row 3

3 – (3 x 0)

3 – 0 = 3

3 – (3 x1)

3 – 3 = 0

0 – (3 x 0)

0 – 0 = 0

0 – (3 x 0)

0 – 0 = 0

1 – (3 x 0)

1 – 0 = 1

0 – (3 x 1)

0 – 3 = -3

0 – (3 x 0)

0 – 0 = 0

1,800 – (3 x 300)

1,800 – 900 = 900

Row 5

-30 – (-40 x 0)

-30 + 0 = -30

-40 – (-40 x 1)

-40 + 40 = 0

0 – (-40 x 0)

0 + 0 = 0

0 – (-40 x 0)

0 – 0 = 0

0 – (-40 x 0)

0 + 0 = 0

0 –(-40 x 1)

0 + 40 = 40

1 – (-40 x 0)

1 + 0 = 1

0 – (-40 x 300)

0 + 12,000

= 12,000

Second Tableau

Second Row

5 – (5 x 1)

5 – 5 = 0

0 – (5 x 0)

0 – 0 = 0

0 – (5 x $\frac{1}{3}$)

0 - $\frac{5}{3}$ = -$\frac{5}{3}$

1 – (5 x 0)

1 – 0 = 1

0 – (5 x 0)

0 – 0 = 0

-8 – (5 x -2)

-8 + 10 = 2

0 – (5 x 0)

0 – 0 = 0

1,600 – (5 x 200)

1600 – 1000 = 600

Row 3

3 – (3 x 1)

3 – 3 = 0

0 – (3 x 0)

0 – 0 = 0

0 – (3 x $\frac{1}{3}$)

0 - $\frac{3}{3}$

0 – 1 = -1

0 – (3 x 0)

0 – 0 = 0

1 – (3 x 0)

1 – 0 = 1

-3 – (3 x -2)

-3 + 6 = 3

0 – (3 x 0)

0 – 0 = 0

900 – (3 x 200)

900 – 600 = 300

Row 4

0 – (0 x 1)

0 – 0 = 0

1 – (0 x 0)

1 – 0 = 1

0 – (0 x $\frac{1}{3}$)

0 – 0 = 0

0 – (0 x 0)

0 – 0 = 0

0 – (0 x 0)

0 – 0 = 0

1 – (0 x -2)

1 – 0 = 1

0 – (0 x 0)

0 – 0 = 0

300 – (0 x 200)

300 – 0 = 300

-30 – (-30 x 1)

-30 + 30 = 0

0 – (-30 x 0)

0 + 0 = 0

0 – (-30 x $\frac{1}{3}$)

0 + $\frac{30}{3}$ = 10

0 – (-30 x 0)

0 + 0 = 0

0 – (-30 x 0)

0 + 0 = 0

40 – (-30 x -2)

40 – 60 = -20

1 – (-30 x 0)

1 + 0 = 1

12,000 – (-30 x 200)

12,000 + 6,000

= 18,000

Third Tableau

Row 1

1 – (-2 x 0)

1 + 0 = 1

0 – (-2 x 0)

0 + 0 = 0

$\frac{1}{3}$ - (-2 x -$\frac{1}{3}$)

$\frac{1}{3}$ - $\frac{2}{3}$

$\frac{1-2}{3}$ = -$\frac{1}{3}$

0 - (-2 x 0)

0 + 0 = 0

0 – (-2 x $\frac{1}{3}$)

0 + $\frac{2}{3}$ = $\frac{2}{3}$

-2 – (-2 x 1)

-2 + 2 = 0

0 – (-2 x 0)

0 + 0 = 0

200 + 200

= 400

0 – (2 x 0) = 0

0 – 0 = 0

0 – (2 x 0)

0 – 0 = 0

-$\frac{5}{3}$ – (2 x -$\frac{1}{3}$)

- $\frac{5}{3}$ + $\frac{2}{3}$

$\frac{-5+2}{3}$ = $\frac{-3}{3}$

= -1

1 – (2 x 0)

1 - 0 = 1

0 – (2 x $\frac{1}{3}$)

0 - $\frac{2}{3}$ = -$\frac{2}{3}$

2 – (2 x 1)

2 – 2 = 0

0 – (2 x 0)

0 – 0 = 0

600 – (2 x 100)

600 – 200 = 400

0 – (1 x 0)

0 – 0 = 0

1 – (1 x 0)

1 – 0 = 1

0 – (1 x -$\frac{1}{3}$)

0 + $\frac{1}{3}$ = $\frac{1}{3}$

0 – (1 x 0)

0 – 0 = 0

1 – (1 x 1)

1 -1 = 0

0 – (1 x 0)

0 – 0 = 0

300 – (1 x 1000)

300 – 100

200

0 – (-20 x 0)

0 + 0 = 0

0 – (-20 x 0)

0 + 0 = 0

+ 10 – (-20 x -$\frac{1}{3}$)

$+ 10 - \frac{20}{3}$	$0 + \frac{20}{3}$	18,000 – (-20 x 100)
$\frac{+30-20}{3} = \frac{10}{3}$	$= \frac{20}{3}$	18,000 + 2,000
0 – (-20 x 0)	-20 – (-20 x 1)	= 20,000
0 + 0 = 0	-20 + 20 = 0	
$0 – (-20 \times \frac{1}{3})$	1 – (-20 x 0)	
	1 + 0 = 1	

Illustration 2:

A company produces three products A, B, C which require processing in two divisions X and Y. the table below shows the times in hours/unit required by these products in the two divisions, and the weekly maximum capacities in hours for the two divisions.

Times required per unit

		Products			Maximum Capacity (hrs/week)
Divisions		A	B	C	
	X	5	3	4	360
	Y	3	2	5	400

The profit margins for A, B, C are ₦16 per unit, N10 per unit, N30 per unit respectively. The company has a contract to supply 60 units of A and /or C in total each week. Determine the quantities of A, B, C which should be produced per week to maximize profit.

Solution:

Let x_1, x_2, x_3 be the quantities of A, B, C respectively produced per week. Then, the linear program is:

To maximize $F = 16x_1 + 10x_2 + 30x_3$ profit

Subject to the constraints: $5x_1 + 3x_2 + 4x_3 \leq 360$ Division A

$3x_1 + 2x_2 + 5x_3 \leq 400$ Division B

$x_1 + x_3 \geq 60$ Contract

$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0.$

The first step is to re-state the problem by converting the constraint inequalities to equations using slack variables S_1, S_2, S_3 and the artificial variable S_4.

To maximize $F = 16x_1 + 10x_2 + 30x_3$

Subject to the constraints: $5x_1 + 3x_2 + 4x_3 = 360$

$3x_1 + 2x_2 + 5x_3 + S_2 = 400$

$x_1 + x_3 – S_3 + S_4 = 60$

$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, S_1 \geq 0, S_2 \geq 0\ S_3 \geq 0, S_4 \geq 0.$

FIRST TABLEAU

x_1	x_2	x_3	S_1	S_2	S_3	S_4	F	Quantity	Ratio
5	3	4	1	0	0	0	0	360	90
3	2	5	0	1	0	0	0	400	80
1	0	1	0	0	-1	1	0	60	60
-16	-10	-30	0	0	0	0	1	0	
1	0	1	0	0	-1	1	0	60	Master Row

SECOND TABLEAU

x_1	x_2	x_3	S_1	S_2	S_3	S_4	F	Quantity	Ratio
1	3	0	1	0	4	-4	0	120	30
-2	2	0	0	1	5	-5	0	100	20
1	0	1	0	0	-1	1	0	60	-60
14	-10	0	0	0	-30	30	1	1,800	
-2	2	0	0	1	5	-5	0	100	Master Row

THIRD TABLEAU

x_1	x_2	x_3	S_1	S_2	S_3	S_4	F	Quantity	Ratio
$\frac{13}{5}$	$\frac{7}{5}$	0	1	$\frac{-4}{5}$	0	0	0	40	
$\frac{-2}{5}$	$\frac{2}{5}$	0	0	$\frac{1}{5}$	1	-1	0	20	
$\frac{3}{5}$	$\frac{2}{5}$	1	0	$\frac{1}{5}$	0	0	0	80	
2	2	0	0	6	0	0	1	2,400	

SECOND TABLEAU

First Row

5 – (4 x 1)

5 – 4 = 1

3 – (4 x 0)

3 – 0 = 3

4 – (4 x 1)

4 – 4 = 0

1 – (4 x 0)

1 – 0 = 1

0 – (4 x 0)

0 – 0 = 0

0 – (4 x -1)

0 + 4 = 4

0 – (4 x 1)

0 – 4 = -4

0 – (4 x 0)

0 – 0 = 0

360 – (4 x 60)

360 – 240 = 120

0 – (-30 x 1)

0 + 30 = 30

2 – (5 x 0)

2 – 0 = 2

5 – (5 x 1)

5 – 5 = 0

0 – (5 x 0)

0 – 0 = 0

1 – (5 x 0)

1 – 0 = 1

0 – (5 x -1)

0 + 5 = 5

0 – (5 x 1)

0 – 5 = -5

0 – (5 x 0)

0 – 0 = 0

400 – (5 x 60)

400 – 300 = 100

Fourth Row

-16 – (-30 x 1)

-16 + 30 = 14

-10 – (-30 x 0)

-10 + 0 = -10

-30 – (-30 x 1)

-30 + 30 = 0

0 –(-30 x 0)

0 – 0 = 0

0 – (-30 x 0)

0 – 0 = 0

0 – (-30 x -1)

0 – 30 = -30

0 – (-30 x 1)

0 + 30 = 30

1 – (-30 x 0)

1 + 0 = 1

0 – (-30 x 60)

0 + 1800 = 1800

THIRD TABLEAU

$1 – (4 \times \frac{-2}{5})$

$1 + \frac{8}{5}$

$\frac{5+8}{5} = \frac{13}{5}$

$3 – (4 \times \frac{2}{5})$

$3 - \frac{8}{5}$

$\frac{15-8}{5} = \frac{7}{5}$

0 – (4 x 0)

0 – 0 = 0

1 – (4 x 0)

1 – 0 = 1

$0 – (4 \times \frac{1}{5})$

$0 - \frac{4}{5} = \frac{-4}{5}$

4 – (4 x 1)

4 – 4 = 0

-4 – (4 x -1)

-4 + 4 = 0

0 – (4 x 0)

0 – 0 = 0

120 – (4 x 20)

120 – 80 = 40

$1 – (-1 \times \frac{-2}{5})$

$1 - \frac{2}{5}$

$\frac{5-2}{5} = \frac{3}{5}$

0 – (-1 x $\frac{2}{5}$)

0 + $\frac{2}{5}$ = $\frac{2}{5}$

1 – (-1 x 0)

1 + 0 = 1

0 – (-1 x 0)

0 – 0 = 0

0 – (-1 x $\frac{1}{5}$)

0 + $\frac{1}{5}$ = $\frac{1}{5}$

-1 –(-1 x 1)

-1 + 1 = 0

1 – (-1 x -1)

1 – 1 = 0

0 – (-1 x 0)

0 – 0 = 0

60 – (-1 x 20)

60 + 20 = 80

-10 – (-30 x $\frac{2}{5}$)

- 10 + 12 = 2

0 – (-30 x 0)

0 + 0 = 0

-30 – (-30 x 1)

- 30 + 30 = 0

1 – (-30 x 0)

1 + 0 = 1

14 – (-30 x - $\frac{2}{5}$)

14 - $\frac{60}{5}$

$\frac{70-60}{5}$

= $\frac{10}{5}$ = 2

Or

14 - $\frac{60}{5}$

14 – 12 = 2

0 – (-30 x 0)

0 + 0 = 0

0 – (-30 x $\frac{1}{5}$)

0 + $\frac{30}{5}$ = 6

30 – (-30 x -1)

30 – 30 = 0

1800 – (-30 x 20)

1800 + 600

= 2,400

MINIZATION PROBLEMS

The simplex algorithm for minimization problems is the same for maximization problems. Minimization problems occur frequently and artificial variables are used to solve them. The problem of minimizing a function is most easily handled by increasing zero variables which correspond to the numerically smallest negative coefficients in the objective row. Actually, for efficiency, we should increase the variable which produces the smallest increase in the objective function, but we opted for the numerically smallest criterion for simplicity.

Illustration:

The linear program is given as:

To minimize $F = 20x_1 + 25x_2$

Subject to the constraints: $4x_1 + 5x_2 \geq 1{,}200$

$x_1 \geq 100$

$2x_1 + 2x_2 \geq 560$

$x_1 \geq 0, x_2 \geq 0.$

You are required to solve the linear program:

i. Graphically
ii. Simplex algorithm

Solution

(a) Using Graphical Method:

i. $4x_1 + 5x_2 = 1{,}200$

When $x_1 = 0$

$5x_2 = 1{,}200$

$x_2 = 240$

Similarly,

When $x_2 = 0$

$4x_1 = 1{,}200$

$x_1 = 300$

when $x_2 = 0$

ii. $x_1 = 100$

iii. $2x_1 + 2x_2 = 560$

When $x_1 = 0$

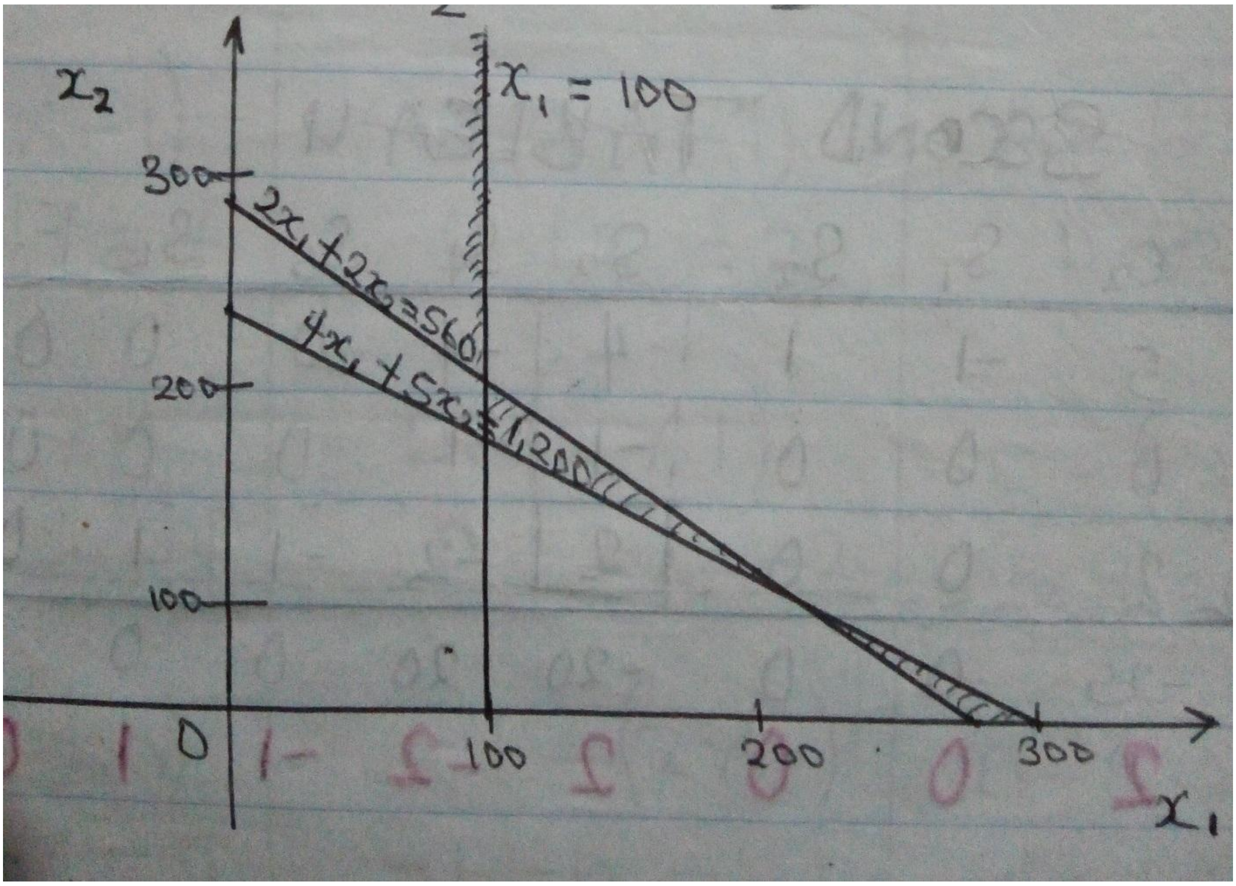
$2x_2 = 560$

$x_2 = \frac{560}{2} = 280$

Similarly, when $x_2 = 0$

$2x_1 = 560$

$x_1 = \frac{560}{2} = 280$



(b) Using simplex algorithm method: i.e

Tableau:

Objective Function

To Minimize: $F = 20x_1 + 25x_2$

Subject to the constraints: $4x_1 + 5x_2 \geq 1{,}200$

$x_1 \geq 100$

$x_1 \geq 0, x_2 \geq 0.$

By introducing slack and artificial variables, the linear program becomes:

To Minimize $F = 20x_1 + 25x_2$

⇨ $F - 20x_1 + 25x_2 = 0$

Subject to the constraints: $4x_1 + 5x_2 – S_1 + S_2 = 1{,}200$

$x_1 – S_3 + S_4 = 100$

$2x_1 + 2x_2 – S_5 + S_6 = 560$

$x_1 \geq 0, x_2 \geq 0, S_1 \geq 0, S_2 \geq 0\ S_3 \geq 0, S_4 \geq 0, S_5 \geq 0, S_6 \geq 0.$

TABLEAU

x_1	x_2	S_1	S_2	S_3	S_4	S_5	S_6	F	Quantity	Ratio
4	5	-1	1	0	0	0	0	0	1,200	300
1	0	0	0	-1	1	0	0	0	100	100
2	2	0	0	0	0	-1	1	0	560	280
-20	-25	0	0	0	0	0	0	1	0	
1	0	0	0	-1	1	0	0	0	100	Master Row

SECOND TABLEAU

x_1	x_2	S_1	S_2	S_3	S_4	S_5	S_6	F	Quantity	Ratio
0	5	-1	1	4	-4	0	0	0	800	200
1	0	0	0	-1	1	0	0	0	100	-100
0	2	0	0	2	-2	-1	1	0	360	180
0	-25	0	0	-20	20	0	0	1	2,000	
0	2	0	0	2	-2	-1	1	0	360	Master Row

THIRD TABLEAU

x_1	x_2	S_1	S_2	S_3	S_4	S_5	S_6	F	Quantity	Ratio
0	1	-1	1	0	0	2	-2	0	80	80
1	1	0	0	0	0	$\frac{-1}{2}$	$\frac{1}{2}$	0	280	280
0	1	0	0	1	-1	$\frac{-1}{2}$	$\frac{1}{2}$	0	180	180
0	-5	0	0	0	0	-10	10	1	5,600	
0	1	-1	1	0	0	2	-2	0	80	Master Row

FOURTH TABLEAU

x_1	x_2	S_1	S_2	S_3	S_4	S_5	S_6	F	Quantity	Ratio
0	1	-1	1	0	0	2	-2	0	80	
1	0	1	-1	0	0	$\frac{-5}{2}$	$\frac{5}{2}$	0	200	
0	0	1	-1	1	-1	$\frac{-5}{2}$	$\frac{5}{2}$	0	100	
0	0	-5	5	0	0	0	0	1	6,000	

SECOND TABLEAU

4 – (4 x 1)

4 – 4 = 0

5 – (4 x 0)

5 – 0 = 5

-1 – (4 x 0)

-1 – 0 = -1

1 – (4 x 0)

1 – 0 = 1

0 – (4 x -1)

0 + 4 = 4

0 – (4 x 1)

0 – 4 = -4

0 – (4 x 0)

0 – 0 = 0

0 – (4 x 0)

0 – 0 = 0

0 – (4 x 0)

0 – 0 = 0

1,200 – (4 x 100)

1,200 – 400 = 800

2 – (2 x 1)

2 – 2 = 0

2 – (2 x 0)

2 – 0 = 2

0 – (2 x 0)

0 – 0 = 0

0 – (2 x 0)

0 – 0 = 0

0 – (2 x -1)

0 + 2 = 2

0 – (2 x 1)

0 – 2 = -2

-1 – (2 x 0)

-1 – 0 = -1

1 – (2 x 0)

1 – 0 = 1

0 – (2 x 0)

0 – 0 = 0

560 – (2 x 100)

560 – 200 = 360

-20 –(-20 x 1)

-20 + 20 = 0

-25 – (-20 x 0)

-25 + 0 = -25

0 – (-20 x 0)

0 + 0 = 0

0 – (-20 x 0)

0 + 0 = 0

0 –(-20 x -1)

0 – 20 = -20

0 – (-20 x 1)

0 + 20 = 20

0 – (-20 x 0)

0 + 0 = 0

0 – (-20 x 0)

0 + 0 = 0

1 – (-20 x 0)

1 + 0 = 1

0 – (-20 x 100)

0 + 2,000 = 2,000

0 – (4 x 0)

0 – 0 = 0

5 – (4 x 1)

5 – 4 = 1

-1 (-4 x 0)

-1 – 0 = -1

1 – (4 x 0)

1 – 0 = 1

4 – (4 x 1)

4 – 4 = 0

-4 – (4 x -1)

-4 + 4 = 0

$0 - (4 \times \frac{-1}{2})$

$0 + \frac{4}{2} = 2$

$0 - (4 \times \frac{1}{2})$

$0 - \frac{4}{2} = -2$

0 – (4 x 0)

0 – 0 = 0

800 – (4 x 180)

800 – 720 = 80

1 – (-1 x 0)

1 + 0 = 1

0 – (-1 x 1)

0 + 1= 1

0 – (-1 x 0)

0 + 0 = 0

0 – (-1 x 0)

0 + 0 = 0

-1 - (-1 x1)

-1 + 1 = 0

1 – (-1 x -1)

1 – 1 = 0

$0 - (-1 \times \frac{-1}{2})$

$0 - \frac{-1}{2} = \frac{-1}{2}$

$0 - (-1 \times \frac{1}{2})$

$0 + \frac{1}{2} = \frac{1}{2}$

0 – (-1 x 0)

0 + 0 = 0

100 – (-1 x 180)

100 + 180 = 280

0 – (-20 x 0)

0 + 0 = 0

-25 – (-20 x 0)

0 + 0 = 0

-25 – (-20 x 1)

-25 + 20 = -5

0 – (-20 x 0)

0 + 0 = 0

-20 – (-20 x 1)

-20 + 20 = 0

20 –(-20 x -1)

20 – 20 = 0

$0 - (-20 \times \frac{-1}{2})$

$0 - \frac{20}{2} = -10$

$0 - (-20 \times \frac{1}{2})$

$0 + \frac{20}{2} = 10$

1– (-20 x 0)

1 + 0 = 1

2,000 - (-20 x 180)

2,000 + 3,600 = 5,600

1 – (1 x 0)

1 – 0 = 1

1 – (1 x 1)

1 – 1 = 0

0 – (1 x -1)

0 + 1 = 1

1 – (1 x 1)

1 – 1 = 0

0 – (1 x 0)

0 – 0 = 0

0 – (1 x 0)

0 – 0 = 0

$\frac{-1}{2}$ – (1 x 2)

$\frac{-1}{2}$ – 2

$\frac{-1-4}{2} = \frac{-5}{2}$

$\frac{1}{2}$ – (1 x -2)

$\frac{1}{2}$ + 2

$\frac{1+4}{2} = \frac{5}{2}$

0 – (1 x 0)

0 – 0 = 0

280 – (1 x 80)

280 – 80 = 200

0 – (1 x 0)

0 – 0 = 0

1 – (1 x 1)

1 – 1 = 0

0 – (1 x -1)

0 + 1 = 1

0 – (1 x 1)

0 – 1 = -1

1 – (1 x 0)

1 – 0 = 1

-1 – (1 x 0)

-1 – 0 = -1

$\frac{-1}{2}$ – (1 x 2)

$\frac{-1}{2}$ – 2

$\frac{-1-4}{2} = \frac{-5}{2}$

$\frac{1}{2}$ – (1 x -2)

$\frac{1}{2}$ + 2

$\frac{1+4}{2} = \frac{5}{2}$

0 – (1 x 0)

0 – 0 = 0

180 – (1 x 80)

180 – 80 = 100

0 – (-5 x 0)

0 + 0 = 0

-5 – (-5 x 1)

-5 + 5 = 0

0 – (-5 x -1)

0 – 5 = -5

0 –(-5 x 1)

0 + 5 = 5

0 – (-5 x 0)

0 + 0 = 0

0 – (-5 x 0)

0 + 0 = 0

-10 – (-5 x 2)

-10 + 10 = 0

10 – (-5 x -2)	1 – (-5 x 0)	5,600 – (-5 x 80)
10 – 10 = 0	1 + 0 = 1	5,600 + 400 = 6,000

Practice Questions

1) Use the simplex algorithm to find the optimum solution for each of the following linear programs:

(a) Maximise: $F = 5x + 20y$

Subject to the constraints: $2x + 6y \leq 300$

$2x + 3y \leq 240$

$y \leq 30$

$x \geq 20$

$y \geq 0$

(b) Maximize: $F = 20x_1 + 10x_2 + 5x_3$

Subject to the constraints: $5x_1 + 2x_2 + 4x_3 \leq 200$

$10x_1 + 6x_2 + 8x_3 \leq 450$

$x_1 \geq 15$

$x_2 \geq 20$

2) Use the simplex algorithm to determine the optimum solution for the linear program

Minimize: $F = 20x + 40y$

Subject to the Constraints: $2x + y \geq 100$

$4x + 3y \geq 240$

$x \geq 0$

$y \geq 20$

3) Use the simplex algorithm to find the optimum solution for the linear program

Minimize: $F = 15x_1 + 30x_2 + 10x_3$

Subject to the constraints: $2x_1 + x_2 + 3x_3 \geq 60$

$4x_1 + 2x_2 + 2x_3 \geq 80$

$2x_1 + 3x_3 \leq 120$

$x_1 \geq 0, x_2 \geq 0, x \geq 0.$

4) An oil company puts additives in its petrol to give improved performance and reduce engine wear. A quantity of petrol has benn ordered and must contain at least 18mgs of additive A and 28 mgs of additive B. The company can use two ingredients X and Y which have the following weights of additives:

Weights of additives in mgs per litre

		ADDITIVE	
		A	B
INGREDIENT	X	1	2
	Y	3	4

For technical reasons, the quantity of petrol ordered must contain at least 3 litres of ingredient X. Both ingredients cost ₦10 per litre.

If the order contains x litre of ingredient X and y litre of ingredient Y, and the cost of the order is to be minimized, formulate a suitable linear program, and use the simplex algorithm to find the least cost solution.

Revision practice with solutions

1) A company manufactures two products P and Q, each of which is processed in three of the company's four divisions. At present, market demand is such that they are able to sell all of their production of P and Q. The processing times in each division for one unit of each product are given below:

Processing times (hours per unit)

Division	Total Processing Time Available in Hours per week
A	360
B	160
C	300
D	140

The revenue per unit is ₦60 per product P and ₦75 for product Q.

(a) If x, y denote the number of units of P and Q respectively produced per week, express the constraints and the weekly revenue function f in terms of x and y.

(b) Write down the initial tableau and the initial feasible solution.

(c) Use the simplex algorithm to determine the quantities of P and Q which should be produced each week to maximize revenue, and state the maximum revenue.

(d) When revenue is maximized, which divisions, if any, have spare capacity?

(e) It the cost of producing one unit of P is ₦15 and one unit of Q is ₦45, calculate the weekly profit when revenue is maximized.

<u>Solution</u>

1. (a) To maximize: $R = 60x + 75y$

Subject to the Constraints:

$$4x + 3y \leq 360$$
$$2x + y \leq 160$$
$$2x + 3y \leq 300$$
$$2x \leq 140$$
$$x \geq 0,\ y \geq 0.$$

(b) The initial Tableau is

X	Y	S_1	S_2	S_3	S_4	F	Quantity	Ratio
4	3	1	0	0	0	0	360	90
2	1	0	1	0	0	0	160	80
2	3	0	0	1	0	0	300	150
2	0	0	0	0	1	0	140	70
-60	-75	0	0	0	0	1	0	

The initial feasible solution is x = 0, y = 0, S_1 = 360, S_2 = 160, S_3 = 300, S_4 =140, F = 0

Second Tableau

X	Y	S_1	S_2	S_3	S_4	F	Quantity	Ratio
2	0	1	0	-1	0	0	60	30
$\frac{4}{3}$	0	0	1	$\frac{-1}{3}$	0	0	60	45
$\frac{2}{3}$	1	0	0	$\frac{1}{3}$	0	0	100	152
2	0	0	0	0	1	0	140	7
-10	0	0	0	258	0	1	7,500	

Third Tableau

X	Y	S_1	S_2	S_3	S_4	F	Quantity	Ratio
1	0	$\frac{1}{2}$	0	$\frac{-1}{2}$	0	0	30	
0	0	$\frac{-2}{3}$	1	$\frac{1}{3}$	0	0	20	
0	1	$\frac{-1}{3}$	0	$\frac{2}{3}$	0	0	80	
0	0	-1	0	1	1	0	80	
0	0	5	0	20	0	1	7,800	

Optimum Solution is $x = 30$ units of P, $y = 80$ units of Q, F = ₦7,800 (where $S_1 = 0$ hours, $S_2 = 20$ hours, $S_3 = 0$ hours, $S_4 = 80$ hours). All figures are per week.

(d) Divisions B and D have spare capacity (20 hours per week and 80 hours per week respectively).

INTEGRATION AND PARTIAL DIFFERENTIATION

Integration and differentiation application can be used to measure rate of changes in business. Integration and differentiation can also be used to find the maximum or minimum point of the curve of certain business functions i.e:

1. Cost functions
2. Profit functions
3. Revenue functions.

It can also be used to determine:

1. Assembly line production can be considered as a function of time or numbers of machines or both.
2. Sales revenue as a function of level of production.
3. Production cost as a function of level of production.
4. Net Present Value as a function of discount rate.

If for a particular business process, the profit is know to be a function of the level of production, then, production level can be determined to bea point that will give maximum profit, i.e

Let $x_1, x_2, x_3, \dots, x_n$ be n variables, then,

$y = f(x_1, x_2, x_3, \dots, x_n)$ as a function of n variables.

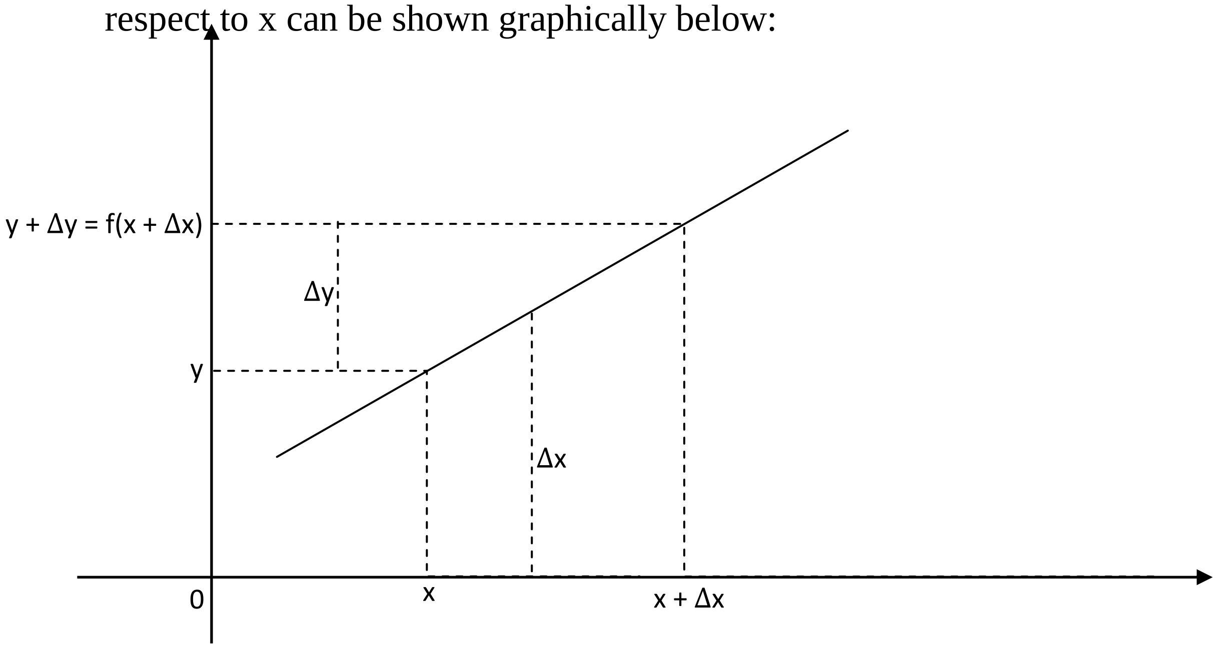
Thus,

y = f(x)

Hence, x is called the independent variable and y is called the dependent variable.

Let y = f(x) be a function.

Then, the differentiation of y measures the rate of change of the function with respect to x can be shown graphically below:



Thus,

Let y = f(x) be any hypothetical function.

Then,

$y + \Delta y = f(x + \Delta x)$.

By making Δy the subject of formula

$\Delta y = f(x + \Delta x) - y$

Since, y = f(x)

Therefore;

$\Delta y = f(x + \Delta x) - f(x)$

By dividing both sides by Δx

Therefore,

$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x} = f(x) - f(x)$

$\frac{\Delta y}{\Delta x} = \frac{\partial y}{\partial x} = y^1 = f^-(x)$

This is called the differentiation of y with respect to x.

Conditions of Derivative Functions:

1. If y is a function of x, then,

$\frac{\partial y}{\partial x}$ is the first derivative of the function.

2. Differentiation can be repeated as many times as necessary on any given function i.e

$\frac{\partial}{\partial x}(\frac{\partial y}{\partial x}) = \frac{\partial^2 y}{\partial x^2}$ is the second derivative function

3. Solving the equation $\frac{\partial y}{\partial x} = 0$, identifies the turning points of a function.

Hence, the turning points of a function is also known as stationary points, are the points where the function f(x) changes sign from negative to positive or from positive to negative.

4. From the second derivative $\frac{\partial^2 y}{\partial x^2}$ can be used to indicate whether the turning point is a maximum or minimum.

   Therefore, to evaluate $\frac{\partial^2 y}{\partial x^2}$ at each (and any) x-value found from $\frac{\partial y}{\partial x} = 0$

i. If $\frac{\partial^2 y}{\partial x^2} > 0$, then the point is minimum

ii. If $\frac{\partial^2 y}{\partial x^2} < 0$, then, the point is maximum

iii. If $\frac{\partial^2 y}{\partial x^2} = 0$, then the point is an inflexional point i.e a mere bend on the curve.

Derivatives of Functions

(a) Basic Method:

If; $y = x^n$

Then, $\frac{\partial y}{\partial x} = nx^{n-1}$

(b) Function with a Constant:

If: $y = x^n + k$ where k is constant. Then,

$\frac{\partial y}{\partial x} = nx^{n-1}$

(c) Function with a Coefficient:

If: $y = ax^n$

Then, $\frac{\partial y}{\partial x} = nax^{n-1}$

(d) Derivatives of a Sum:

Let: U = g(x) and V = h(x)

Thus, f(x) = U + V

Then;

$\frac{\partial}{\partial x}(U + V)$

$= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}$

(e) Derivative of a Product:

Suppose: U and V are functions of x. If U = g(x), V = h(x) and y = UV.

Then, $\frac{\partial}{\partial x}(UV) = \frac{U\,\partial V}{\partial x} + \frac{V\,\partial U}{\partial x}$

(f) Derivative of a Quotient:

If: $y = \frac{U}{V}$

Then, $\frac{\partial y}{\partial x} = \frac{\frac{V\,\partial U}{\partial x} - \frac{U\,\partial V}{\partial x}}{V^2}$

Illustration 1:

Find $\frac{\partial y}{\partial x}$ and $\frac{\partial^2 y}{\partial x^2}$ and determine the coordinates and nature of the turning points of the function;

$y = x^3 – 7.5x^2 + 18x + 6$

Solution:

$\frac{\partial y}{\partial x} = 3x^2 – 15x + 18$

Then, from $\frac{\partial y}{\partial x} = 0$, gives

$3x^2 – 15x + 18 = 0$

By dividing all through 3 i.e $\frac{3x^2}{3} - \frac{15x}{3} + \frac{18}{3} = 0$

$\therefore x^2 – 5x + 6 = 0$

By factorization; (x – 2) (x – 3) = 0

Therefore,

x = 2; x = 3

by substituting $x = 2$

therefore,

$y = x^3 – 7.5x^2 + 18x + 6$

$= 2^3 – 7.5(2)^2 + 18(2) + 6$

$= 8 – 30 + 36 + 6$

$= -30 + 50$

$\therefore y = 20$

Similarly, by substituting $x = 3$

$y = x^3 – 7.5x^2 + 18x + 6$

$3^3 – 7.5(3)^2 + 18(3) + 6$

$= 27 – 67.5 + 54 + 6$

$= -67.5 + 87$

$\therefore y = 19.5$

Therefore, the coordinates are:

(2, 20) ; (3, 19.5)

$\frac{\partial y}{\partial x} = 3x^2 – 15x + 18$

$\frac{\partial^2 y}{\partial x^2} = 6x – 15$

When $x = 2$

$\therefore \frac{\partial^2 y}{\partial x^2} = 6x – 15$

$= 6(2) – 15$

$= 12 – 15$

$= -3$

$\therefore y = -3 \Longrightarrow$ and thus signifies a maximum

When $x = 3$

$\therefore \frac{\partial^2 y}{\partial x^2} = 6x – 15$

$= 6(3) – 15$

$= 18 – 15$

$= 3$

$\therefore y = 3 \Longrightarrow$ and thus signifies a minimum.

∴ For (2, 20) is maximum point

(3, 19.5) is minimum point

Illustration 2:

A food processing plant has a particular problem with delivery and processing of perishable goods with a short life. All deliveries must be processed in a single day and although there are a number of processing machines available, they are very expensive to run. A researcher has developed the function: $y = 12x – 2a – ax^2$ to describe the profit (y in ₦'000) given the number of machines used (x) and the number of deliveries (a) in a day.

i. Show that the system is uneconomic if 4 deliveries are made in a day (i.e a = 4).
ii. If three deliveries are made in one day, find the number of processing machines that should be used in order that the profit is maximized. In this case, what is the maximum profit?

Solution:

i. $y = 12x – 2a – ax^2$

when a = 4, we have

$y = 12x – 2(4) – 4x^2$

$\therefore 12x – 8 – 4x^2$

Note:

An uneconomic system will result if the profit is not positive no matter how many processing machines are used i.e for all values of x.

Since, $y = 12x – 8 – 4x^2$

$\frac{\partial y}{\partial x} = 12 – 8x$

Moreso, $\frac{\partial^2 y}{\partial x^2} = -8$; this signifies a maximum.

Since, $\frac{\partial y}{\partial x} = 0$,

$12 – 8x = 0$

$12 = 8x$

$\therefore x = 1.5$

Note:

Since the number of machines must be a whole number not in fraction or decimal, therefore, we have to show that with either 1 or 2 machines, the profit made is positive.

Hence,

$y = 12x – 8 – 4x^2$

When x = 1

$y = 12(1) – 8 – 4\ (1)^2$

$= 12 – 8 – 4$

$\therefore y = 0$

When x = 2

$y = 12x – 8 – 4x^2$

$= 12(2) – 8 – 4(2)^2$

$= 24 – 8 – 16$

$\therefore y = 0$

Therefore, in both cases, there is no profit. Thus, the system is uneconomic if 4 deliveries are made.

ii. When a = 3

$y = 12x – 2a – ax^2$

$y = 12x – 2(3) – 3x^2$

$= 12x – 6 – 3x^2$

$\frac{\partial y}{\partial x} = 12 – 6x$

$\frac{\partial^2 y}{\partial x^2} = -6$; this signifies a maximum.

Hence, $\frac{\partial y}{\partial x} = 0$

Thus, when x = 2

$y = 12x – 6 – 3x^2$

$= 12(2) – 6 – 3(x)^2$

$= 24 – 6 – 12$

$= 24 – 18$

$\therefore y = 3$ (₦'000)

Therefore, using 2 processing machines, the profit is ₦6,000.

Practice Questions

Differentiate each of the following functions and determine the coordinates and nature of the turning points;

i. $y = 3x^2 – 210x + 20$
ii. $y = 2x^2 – 4x + 10$
iii. $y = 4x^2 – 12x + 16$
iv. $y = 3x^2 + 4x – 8$
v. $y = 12x^2 – 10x + 30$
vi. $y = 2x^3 – 5x^2 + 14$

COST, REVENUE AND PROFIT FUNCTIONS

COST FUNCTIONS:

The cost involved in standard processes can be normally categorized as follows:

(a) Fixed Costs: These are costs that need to be borne before production can physically begin and are independent of the;

i. The purchase of equipment
ii. The rent of equipment
iii. The lease of equipment
iv. Fixed overheads

v. Transportation cost

vi. Manpower movement costs

(b) Variable Costs: These depend on the number of items to be produced. The costs are associated with;

i. The supply of raw materials

ii. The overheads necessary to manufacture each product

(c) Special Costs: These are optional costs factors and are sometimes included in the total costs function.

These cover costs relating to:

i. Storage

ii. Maintenance

iii. Deterioration

FORM OF A COST FUNCTION

A total cost function takes the general form:

$C(x) = a + bx + cx^2$

Where:

x - is the quantity of items demanded, supplied or produced

a - is the fixed cost associated with the product

b - is the variable cost per item

c - is the (optional) special cost factor and is relatively small.

Illustration

The variable costs associated with a certain process are ₦0.65 per item. The fixed costs per day have been calculated as ₦250 with special costs estimated at ₦$0.02x^2$, where x is the size of the production run i.e number of items produced.

(a) Derive a function to describe cost per item for a day's production.

(b) Calculate the size of the daily run that will minimize cost per item.

(c) Find the cost of a day's production for a run that minimize cost per item.

Solution

Cost per item = $\frac{TC}{x}$

(a) The total cost function:

$TC = a + bx + cx^2$

Where: a = ₦250, b = ₦0.65 per item c = ₦$0.02x^2$

∴ $TC = 250 + 0.65x + 0.02x^2$

For the production of x items

∴ the cost per item

$C(x) = \frac{TC}{x} = \frac{250+0.65x+0.02x^2}{x}$

That is: $\frac{250}{x} + \frac{0.65x}{x} + \frac{0.02x^2}{x}$

Thus, $C(x) = \frac{250}{x} + 0.65 + 0.02x$

∴ $C(x) = \frac{250}{x} + 0.65 + 0.02x$

(b) The cost per item that will be minimized: that is, when

$\frac{\partial c}{\partial x} = 0$, gives

Since $C(x) = \frac{250}{x} + 0.65 + 0.02x$

∴ $\frac{\partial c}{\partial x} = \frac{250}{x} + 0.65 + 0.02x$

$= 250x^{-} + 0.65 + 0.02x$

$\frac{\partial c}{\partial x}$ = 250x(-1)x^{-1-1} + 0.02

∴ -250x^{-2} + 0.02 = 0

-250x^{-2} = -0.02

By dividing both sides by negative (-)

∴ 250x^{-2} = 0.02

Which can also be written as:

$\frac{250}{x^2}$ = 0.02

By cross multiplication, thus

0.02x^2 = 250

By dividing both sides by 0.02

i.e $\frac{0.02x^2}{0.02} = \frac{250}{0.02}$

x^2 = 12,500

by taking the square roots of both sides

$x^2 = \sqrt{12,500}$

∴ x = 112 items

∴ x = 112 which minimizes the cost per item

(c) Since TC = 250 + 0.65x + 0.02x^2 therefore, when x = 112

TC = 250 + 0.65(112) + 0.02$(112)^2$

= 250 + 0.65(112) + 0.02(12,544)

= 250 + 72.8 + 250.88

= 573.68

∴ the cost of a day's run which minimizes cost per item is ₦573.68

REVENUE FUNCTIONS:

In a real business environment, it is quite usual for items prices to depend on the number of items in demand. That is, the more items that are in demand, the less the price per unit. hence, the price $P_{(r)}$ is a function of x. Thus,

P_r (x) is known as the demand function and the revenue function takes the form $R(x) = x \cdot P_r(x)$

Where:

x – is the quantity of items demanded, supplied or produced.

P_r (x) is the fixed cost associated with the product

Illustration:

Given the demand function: $P_r(x) = 10.4 – 1.3x$ where x is in hundreds, find the level of production (i.e the value of x) which minimizes total revenue.

Solution

The revenue function is $R(x) = x \cdot P_r(x)$.

Since, $P_r(x) = 10.4 – 1.3x$

∴ $R(x) = x(10.4 – 1.3x)$

$= 10.4x – 1.3x^2$

∴ $R(x) = 10.4x – 1.3x^2$

To maximize revenue, we solve $\frac{\partial R}{\partial x} = 0$

$\therefore \frac{\partial R}{\partial x} = 10.4x - 1.3x^2$

$= 10.4 - 2.6x$

$\therefore 2.6x = 10.4$

By dividing both sides by 2.6

$\therefore x = 4$

Thus, $\frac{\partial^2 R}{\partial x^2} = 10.4 - 2.6x$

= -2.6 signifies a maximum.

∴ the level of production which maximizes the revenue is 400 units i.e (4 x 100).

PROFIT FUNCTIONS

Let x be the quantity of items demanded, supplied or produced.

C(x) – is the cost function in terms of x involved in the production

R(x) – is the revenue function in terms of x obtained from the sale of a number of products.

P(x) – is the profit function in terms of x.

$\therefore P(x) = R(x) - C(x)$

To find the maximum profit point for any process, we solve the equation

$\frac{\partial P}{\partial x} = 0$ for x

$\therefore \frac{\partial^2 P}{\partial x^2} < 0$ at x value obtained gives the maximum profit point.

Thus, P(x) = R(x) – C(x) at that value of x which gives the maximum profit.

Illustration:

A manufacturer known that if x (hundred) products are demanded in a particular week;

i. The total cost function (₦'000) is 14 + 3x
ii. The total revenue function (₦'000) is $19x - 2x^2$

You are required to:

(a) Derive the (total) profit function
(b) Find the profit break-even points
(c) Calculate the level of demand that maximizes the profit and the amount of profit obtained.

Solution

(a) C(x) = 14 + 3x

$R(x) = 19x - 2x^2$

∴ the total profit function P(x) is:

$P(x) = R(x) - C(x)$

$= 19x - 2x^2 - (14 + 3x)$

$= 19x - 2x^2 - 14 - 3x$

By collecting the likes terms:

$= 19x - 3x - 2x^2 - 14$

$\therefore P(x) = 16x - 2x^2 - 14$

(b) For break-even points, we have:

P(x) = 0

$\therefore P(x) = 16x - 2x^2 - 14$

$\therefore 16x - 2x^2 - 14 = 0$

By dividing all through by 2

i.e $\frac{16x}{2}$ - $\frac{2x^2}{2}$ - $\frac{14}{2}$ = 0

= $8x - x^2 - 7 = 0$

We can rewrite it as:

$-x^2 + 8x - 7 = 0$

By multiplying all through by negative (-)

$\therefore x^2 - 8x + 7 = 0$

By factorization:

$\therefore (x - 1)(x - 7) = 0$

$\therefore x = 1 ; x = 7.$

Therefore, the profit break-even points are when the demand is 100 or 700 units products.

(c) $P(x) = 16x - 2x^2 - 14$

$\frac{\partial P}{\partial x} = 16 - 4x$

$\therefore \frac{\partial^2 P}{\partial x} = -4$ signifies maximum

Thus, $\frac{\partial P}{\partial x} = 0$ gives x = 4 i.e the maximum profit point

Since, $P(x) = 16x - 2x^2 - 14$

When x = 4

$P(x) = 16(4) - 2(4)^2 - 14$

$= 64 - 32 - 14$

$= 64 - 46$

$\therefore P(x) = 18$

Hece, P is in (₦'000)

$\therefore$ 18 x 1000 = 18,000.

$\therefore$ the maximum profit is ₦18,000 when the weekly demand is 400 units (i.e x = 4, that is, 4 x 100 = 400).

DEMAND (PRICE) FUNCTION

Price or (demand) function are used to vary the price of an item according to how many items are being considered. Thus, the more the number of items, the less the price per item and vice-versa. This is simply the standard business principle at the economy of scale where it is generally more efficient to operate on a large scale as can be coped with.

The price or demand functions are normally linear and take the form:

$P_r(x) = a + bx$

Where:

x – is the quantity of items demanded, supplied or produced.

a, b – are coefficients that can take any numeric values

$P_r(x)$ – is the price function.

PROCEDURE FOR OBTAINING A PRICE OR DEMAND FUNCTION

Suppose we are given the following information:

i. $P_r(x) = a + bx$ i.e linear function
ii. When the price of an item is P_1 and x_1 items are demanded
iii. When the price is P_2 and x_2 items are demanded.

If we substitute the information (ii), (iii) and in (i), then, we have two equations: i.e

$P_1 = a + bx_1$

$P_2 = a + bx_2$

Hence, solving the two equations above gives the value of (a) and (bx) from which we can obtain the demand function.

Illustration:

Given that the price of an item as ₦3.50 when 250 items are demanded but when only 50 are demanded, the price rises to ₦5.50 per item, identify the linear demand function and calculate the price per item at a demand level of 115.

Solution:

(a) $P_r (x) = a + bx$

Firstly: By substituting for $P_r = 3.5$ and $x = 250$, which gives:

3.5 = a + 250b and

Secondly: By substituting for $P_r = 5.5$ and $x = 50$, which gives:

5.5 = a + 50b

Therefore,

P_1 gives 3.5 = a + 250b ------------ eq (i)

P_2 gives 5.5 = a + 50b -------------- eq (ii)

By subtracting equ (ii) from equ (i) gives

5.5 – 3.5 = a – a + 50b – 250b

= 2 = -200b

By dividing both sides by -200

:- $\frac{2}{-200} = \frac{-200b}{-200}$

$\therefore$ b = -0.01

Therefore, by substitution b = -0.01 in equ(i) or (ii), gives

3.5 = a + 250 (-0.01)

3.5 = a – 2.5

By collecting the like terms:

$\therefore$ 3.5 + 2.5 = a

Thus, a = 6, b = -0.01

Therefore, the demand function $P_r = 6 - 0.01x$

(b) Hence: When x = 115

By substituting x = 115 in the demand function;

Thus,

$P_r = 6 - 0.01(115)$

= 6 – 1.115

$\therefore P_r = 4.85$

Therefore, at a demand level of 115, the price is ₦4.85 per item.

MARGINAL COST AND REVENUE FUNCTIONS

i) Marginal Cost Function: This can be interpreted as the extra cost incurred of producing another item of activity (demand) level of x. If C(x) is the total cost function for some process, then;

$\frac{\partial c}{\partial x}$ is defined as the marginal cost function

ii) Marginal Revenue Function: This can be interpreted as the extra revenue obtained from producing another item at activity level of x. if R(x) is the revenue function for some process, then:

$\frac{\partial R}{\partial x}$ is defined as the marginal revenue function.

i. **Maximum Profit Point:** The maximum profit point for some process can be found by solving the equation: $\frac{\partial R}{\partial x} = \frac{\partial c}{\partial x}$

Illustration:

A car component manufacturer can sell all the components of a particular type that he can produce. The total cost (₦) of producing q components per week is given by: 300q + 2000.

The demand function (₦) is estimated as 500 – 2q.

(a) Derive the revenue function R.
(b) Obtain the total profit function
(c) How many units per week should be produced in order to maximize profit?
(d) Show that the solution of the equation $\frac{\partial R}{\partial x} = \frac{\partial c}{\partial x}$ where C represents the function which gives the same value for q as in part (c).
(e) What is the maximum profit available?

Solution:

(a) To derive revenue function:

P_r (q) = 500 – 2q

Revenue function: R = q . P_r(q)

= q (500 – 2q)

Revenue Function (R) = 500q – $2q^2$

(b) Total profit function: P(q) = R(q) – C(q)

Where:

R(q) = 500q – $2q^2$

$C(q) = 300q + 2000$

$\therefore P(q) = R(q) – C(q)$

Therefore, $500q – 2q^2 – (300q + 2000)$

$= 500q – 2q^2 – 300q – 2000$

By collecting the like terms:

$= 500q – 300q – 2q^2 – 2000$

$=200q – 2q^2 – 2000$

$\therefore P(q) = 200q – 2q^2 – 2000.$

(c) $\frac{\partial P}{\partial q} = 0$

$P(q) = 200q – 2q^2 – 2000$

$\therefore$ for $\frac{\partial P}{\partial q} = 0$

Then, $\frac{\partial P}{\partial q} = 200 – 4q$

Therefore, $200 – 4q = 0$

By dividing both sides by 4

$\therefore q = 50$ units.

$\frac{\partial^2 P}{\partial q^2} = -4$ signifies maximum point

$\therefore q = 50$ units per week should produce maximum profit.

(d) $\frac{\partial R}{\partial q} = \frac{\partial c}{\partial q}$

Since, $\frac{\partial R}{\partial q} = 500q – 2q^2$

= 500 – 4q

Similarly, $\frac{\partial c}{\partial q}$ = 300q + 2000

= 300

Since, $\frac{\partial R}{\partial q} = \frac{\partial c}{\partial q}$ which gives 500 – 4q = 300

By collecting thelike terms

-4q = 300 – 500

-4q = -200

By dividing both sides by (-) negative

4q = 200

∴ q = 50 units as in (c)

(e) P(q) = 200q – $2q^2$ – 2000

Where q = 50

∴ P(q) = 200 (50) – $2(50)^2$ – 2000

= 10,000 – 2(2,500) – 2000

= 10,000 – 5,000 – 2,000

∴ P(q) = ₦3,000

Therefore, the maximum profit is ₦3,000.

INTEGRATION

Integration can be regarded as the opposite process to differentiation. For instance, consider the function of f(x). The integral of f(x) with respect to x is written as: $\int f(x)\partial x$

Types of Integrals:

(a) Indefinite Integral: For instance:

If y = f(x), the indefinite integral of f(x) with respect to x is written as:

$$\int f(x)\partial x$$

(b) Definite integral: The definite integral is written as:

$\int_a^b f\ (x)\partial x$ Where b is the upper limit and a is the lower limit

Illustration Under Indefinite Integral:

Evaluate: y = $5x^2$ + 3x + 4.

Solution:

$\int y\ \partial x$ = $\int 5x^2 + 3x + 4)\ \partial x$

= $\int 5x^2 \partial x + \int 3x\partial x + \int 4\ \partial x$

= $\frac{5x^3}{3}$ + $\frac{3x^2}{2}$ + 4x + c

Where c = (is constant).

Illustration:

Evaluate:

y = 5x + 4

Solution:

$$\int_1^4 (5x + 4)\partial x$$

$= (\frac{5x^2}{2} + 4x)_1^4$

$= [\frac{5}{2}(4) + 4(4)] – [\frac{5}{2}(1)^2 + 4(1)]$

$= [\frac{5}{2}(16) + 16] – [\frac{5}{2} + 4]$

$= 56 - 6\frac{1}{2}$

$\therefore y = 49\frac{1}{2}$

PRACTICAL USE OF INTEGRATION

Integration can be used in business environment to find:

i. Revenue function given marginal revenue function

ii. Cost function given marginal cost function.

Illustration:

The total revenue obtained (in ₦'000) from selling x hundred items in a particular day is given by R which is a function of variable x.

Given $\frac{\partial R}{\partial x} = 20 – 4x$

$\int \frac{\partial R}{\partial x} = \int (20 - 4x)\partial x$

$\therefore R = 20x – 2x^2 + c$

Where x = 0, thus, the revenue is 0.

$\therefore 0 = 20(0) – 2(0)^2 + c$ where c = 0

∴ Total Revenue Function (R) = $20x – 2x^2$

b) To find x that maximizes the revenue: $\frac{\partial R}{\partial x}$ = 0, which gives

i.e $20x – 2x^2 = 0$

∴ $20 – 4x = 0$

$-4x = -20$

By dividing both sides by negative (-)

$4x = 20$

By dividing both sides by 4

∴ x = 5 (in hundreds) i.e x = 500.

Therefore, to determine total revenue when x = 5.

Total Revenue = $20x – 2x^2$

∴ $R = 20(5) – 2(5)^2$

$= 100 – 50$

= 50 (₦'000)

Therefore, total revenue (R) = ₦50,000.

<u>Practice Questions:</u>

1. The profit (in ₦'000) from a daily production run is given by P, which is a function of the level of production x (in '000). If $\frac{\partial P}{\partial x}$ = 11 – 2x and one profit break-even point is known to be a production of 3,000. Find:
   (a) P as a function of x.
   (b) The other profit break-even point

(c) The daily production run that gives the maximum profit

(d) The value of the maximum daily profit.

2. Your company has recently started to give economic advice to your clients. Acting as a consultant you have estimated the demand curve of a clients company to be: AR = 200 – 8x. Where AR is average revenue (₦). further investigation has shown that the company's costs when not producing output are ₦10.

   (a) Find the equation of total cost

   (b) If total revenue is average revenue multiplied by output, find the equation of total revenue.

   (c) Find the turning points of the company's profit curve and say whether these are maxima or minima.

   (d) Find the equation of marginal revenue.

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ii. Business Mathematics and Statistics by: Andne Frances.

iii. Modern Microeconomics: Second Edition by Koyutsoyiannis, A.

iv. Suggested further reading in the class